**Efficient Modeling and Representation of Agreement in Interval-Valued Data**

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**Abstract**—Recently, there has been much research into effective representation and analysis of uncertainty in human responses, with applications in cyber-security, forest and wildlife management, and product development, to name a few. Most of this research has focused on representing the response uncertainty as intervals, e.g., “I give the movie between 2 and 4 stars.” In this paper, we extend upon the model-based interval agreement approach (IAA) for combining interval data into fuzzy sets and propose the efficient IAA (eIAA) algorithm, which enables efficient representation of and operation on the fuzzy sets produced by IAA (and other interval-based approaches, for that matter). We develop methods for efficiently modeling, representing, and aggregating both crisp and uncertain interval data (where the interval endpoints are intervals themselves). These intervals are assumed to be collected from individual or multiple survey respondents over single or repeated surveys; although, without loss of generality, the approaches put forth in this paper could be used for any interval-based data where representation and analysis is desired. The proposed method is designed to minimize loss of information when transferring the interval-based data into fuzzy set models and then when projecting onto a compressed set of basis functions. We provide full details of eIAA and demonstrate it on real-world and synthetic data.

**I. INTRODUCTION**

Surveys have long been a powerful way to elicit information and knowledge from groups of individuals and, recently, crowd-sourcing—surveying large groups of people to solve problems—has become a very popular research topic. The information has many uses, from analysis of results to inference of conclusions, relationships, or decision-making dynamics. Recently, there has been much interest in employing survey tools that enable the capture of response uncertainty with special emphasis on the creation of expert and decision-support systems that take uncertainty into account [1–4]. Fuzzy logic, in particular, has been found to be effective for analysis of survey results, with Type-2 (T2) fuzzy logic and, the related, Computing with Words paradigm leading the pack [1, 3–7].

The aforementioned uncertainty in survey data comes from many factors, but, in particular, two main contributors that we address are i) inter-expert uncertainty: the variation among the individual opinions in a group of participants; and ii) intra-expert uncertainty: the variation (or vagueness) in the opinions of individual participants - usually over time. While intra-expert uncertainty can come from either dynamic uncertainty—the respondent changes his answer when repeatedly queried—or indecision—the respondent is unsure about their response—we will, in this paper, simply assume that survey responses come in one of two forms: i) a T1 interval or ii) a T2 interval. Figure 1 shows two examples of what a T1 interval response looks like. In view (a), a fairly certain interval-response is shown, while view (b) shows a more uncertain interval.

[Figure 1: Example of interval responses from a real survey, where (a) is more certain than (b). (from [8])]
in the input set of intervals—often the difference of 100s or 1,000s of endpoints versus 10s of weights. This not only produces an efficient representation of the FS, but also reduces the computational complexity of FS operations further down the analysis pipeline.

This paper is organized as follows. Section II provides a brief background on the topics addressed in this paper, while Section III describes how sets of intervals can be aggregated into FSs—namely, the IAA method. Section IV proposes the efficient IAA (eIAA) method and Section V demonstrates eIAA on some real survey data. Finally, Section VI briefly summarizes this paper. Select acronyms and notation are contained in Table I.

II. BACKGROUND

In this section we provide a very brief introduction to the interval and FS models used in this paper [9]. For a more thorough treatment of these topics, please see refs. [8–10].

A. Crisp and Uncertain Intervals

Two types of intervals are considered in this paper: i) crisp intervals, and ii) uncertain intervals [8]. These intervals are shown in Fig. 2. Crisp intervals are denoted as \( A = [l_A, r_A] \), where \( l_A \) is the left endpoint and \( r_A \) is the right endpoint. Uncertain intervals are denoted as \( A = [l_A, r_A] \), where the left and right endpoints are intervals themselves: \( l_B = [l_B, r_B] \) and \( r_B = [l_B, r_B] \).

B. Interval Type-2 Fuzzy Sets

GT2 FSs [9] were first introduced by Zadeh [11] as a generalization of T1 FSs. In these sets, the membership \( \mu \) at each point on the domain \( x \) is itself defined as a T1 FS. GT2 FSs introduce a burdensome computational complexity and thus many representations, simplifications, and approximations have been introduced to make application of GT2 FSs tractable [10, 12–14]. One of these is the IT2 FS [15], where the T2 FS is considered to have a secondary membership that is 1 on an interval \([\bar{y}(x), y(x)\])

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
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<tbody>
<tr>
<td>( \mu_A(x) )</td>
<td>Membership of ( A ) at value ( x )</td>
</tr>
<tr>
<td>( A = [l_A, r_A] )</td>
<td>Interval between values ( l ) and ( r ) (see Fig. 2a)</td>
</tr>
<tr>
<td>( \bar{A} = [l_A, \bar{r}_A] )</td>
<td>Uncertain interval between values ( l = [l_1, r_1] ), and ( \bar{r} = [\bar{l}_1, \bar{r}_1] ) (see Fig. 2b)</td>
</tr>
<tr>
<td>( i = [m] )</td>
<td>The set of integers, ( i = 1, 2, \ldots, n )</td>
</tr>
<tr>
<td>( i = [a : b] )</td>
<td>The set of integers, ( i = a, a+1, \ldots, b )</td>
</tr>
<tr>
<td>( 1_m )</td>
<td>( m )-length vector of ones</td>
</tr>
<tr>
<td>( 0_m )</td>
<td>( m )-length vector of zeroes</td>
</tr>
<tr>
<td>( \lambda(x) )</td>
<td>Linguistic basis function, ( \lambda : x \rightarrow [0, 1] ) (see Definition 1)</td>
</tr>
<tr>
<td>( \phi(x) )</td>
<td>General basis function, ( \phi : x \rightarrow \mathbb{R} )</td>
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and the membership function that follows \( y(x) \) is the lower membership function (LMF), where both the UMF and LMF are T1 FSs [16].

Commonly, the membership function of a T1 FS is noted as \( \mu \), while for T2 FSs the primary membership is denoted \( u \) and the secondary membership is denoted \( \mu \) [16]. The primary membership is on the \( y \), or vertical, axis and the secondary membership is on the \( z \), or “towards-you,” axis. An IT2 FS is fully described by its UMF and LMF, where the IT2 FS \( A \) has the (secondary) membership function that has values of 1 between the UMF and LMF.
that contain that value. We can simplify (1) for computational purposes in two different ways. First, for any value \( x \), we can simply count the number of intervals in which \( x \) resides, computing the membership as

\[
\mu_A(x) = \frac{1}{n} \left( \sum_{i=1}^{n} l_{A_i} \leq x \right) \text{ or } \mu_A(x) = \frac{1}{n} \left( \sum_{i=1}^{n} r_{A_i} \leq x \right),
\]

(2a)

\[
\mu_{A_i}(x) = \begin{cases} 
1 & l_{A_i} \leq x \leq r_{A_i}, \\
0 & \text{else}. 
\end{cases}
\]

(2b)

The second simple formulation is convenient if one wishes to compute all the locations in the membership function where transitions from one level \( y_i \) to either \( y_{i-1} \) or \( y_{i+1} \) occur. It is easy to show that the membership function can be calculated as

\[
\mu_A(x) = \frac{1}{n} \left( \sum_{i=1}^{n} (l_{A_i} \leq x) - \sum_{i=1}^{n} (r_{A_i} \leq x) \right),
\]

(3)

where this gives the membership of \( A \) at \( x \) by counting the number of left endpoints in \( A \) less than \( x \) and subtracting the number of right endpoints in \( A \) less than \( x \). Thus, let \( X_A = \{x_1, \ldots, x_{2n}\} \) be the set of sorted interval endpoints, where \( x_1 \leq x_2 \leq \ldots \leq x_{2n} \), and let \( B = \{b_1, \ldots, b_{2n}\} \) be an associated set of indicator variables, where \( b_i = +1 \) if \( x_i \) is a left interval endpoint and \( b_i = -1 \) if \( x_i \) is a right interval endpoint. Then,

\[
u_A(x_i) = \frac{1}{n}, \quad i = 1 : 2n.
\]

(4a)

\[
u_A(x_i) = u_A(x_{i-1}) + b_i/n, \quad i = 2 : 2n.
\]

(4b)

Using this formulation, one can build up the membership function by looping through \( X \), from \( x_1 \) to \( x_{2n} \). The membership at any other value \( x \) is simply

\[
u_A(x) = \begin{cases} 
u_A(x_i) & x_i \leq x < x_{i+1}, \quad i = 2n - 1, \\
0 & \text{else}. \end{cases}
\]

(5)

When the intervals are presented with uncertain endpoints, IAA produces IT2 FSs, which we describe next.

**B. Uncertain intervals \( \tilde{A} \)**

Let \( \tilde{A} = \{\tilde{A}_1, \ldots, \tilde{A}_N\} \) be a set of uncertain intervals, where the interval endpoints are represented as having interval uncertainty themselves. To combine uncertain intervals, we proceed similarly as for crisp intervals; however, we apply (1) independently for all “outer” and “inner” endpoints, resulting in a T1 UMF and LMF, respectively, of an IT2 FS. Let \( \tilde{A}_i = [l_i, r_i] \) notate the \( i \)th uncertain interval in \( \tilde{A} \). Also, let the outermost (most uncertain) set of intervals in \( \tilde{A} \) be \( \tilde{A}_{U} = \{ [l_{i_1}, r_{i_1}], \ldots, [l_{i_n}, r_{i_n}] \} \), and the innermost (least uncertain) set of intervals in \( \tilde{A} \) be \( \tilde{A}_{L} = \{ [r_{i_1}, l_{i_1}], \ldots, [r_{i_n}, l_{i_n}] \} \). Then, mathematically, the following operations produce the IT2 FS \( \tilde{A} \) that models \( \tilde{A} \).

\[
\text{UMF}(\tilde{A}) = \text{IAA}(\tilde{A}_{U}); \quad \text{LMF}(\tilde{A}) = \text{IAA}(\tilde{A}_{L}).
\]

(6)

Note that the second efficient formulation of \( \mu_{A} \) considers the intervals to be \( A_i = [l, r] \), where the right endpoint is not included in the interval.

**C. Computational Complexity of IAA**

While IAA has been shown to be effective in representing interval survey data [8], a potential weakness is in the way it represents the FSs which model the combined interval data—in particular when a high number of intervals are considered. For crisp intervals, the resulting T1 FS that represents \( n \) intervals is be modeled by \( (at most) 2n \) pairs of domain values and the membership at that value, \( \{X, \mu(X)\}^{2n} \)—assuming unique interval endpoints for the survey responses—where (5) is used to determine the membership from these pairs. For uncertain intervals both the UMF and LMF must be modeled, resulting in \( 4n \) pairs. While this is not unreasonable for modern storage, this can cause FS operations on these models to be computationally expensive.

Consider two T1 FSs as modeled by IAA, \( A \rightarrow \{X, \mu_A(X)\}^{2n} \) and \( B \rightarrow \{Y, \mu_B(Y)\}^{2n} \), where \( A \) represents \( n \) intervals and \( B \) represents \( m \) intervals. To compute the operation \( A \cup B \), one can apply (7) on the sorted sets produced by (5). This, however, can result in a union set that, in the worst case, is represented as \( 4nm \) pairs, \( \{X \cup Y, \mu_{A \cup B}(X \cup Y)\}^{4nm} \). Furthermore, computing (7) costs \( O(n + m) \) computations: the cost of sorting \( z \) into \( X \) and \( Y \). In the next section, we will show how an approximation based on a linear-constraint least squares problem can reduce this cost.

**IV. EFFICIENT IAA**

Consider a FS as modeled by IAA, \( A \rightarrow \{X, \mu_A(X)\}^{2n} \). The eIAA method compresses the membership function \( \mu_A(x) \) by representing it as the weighted sum of a set of basis functions. We propose two different types of basis sets and methods for learning the weights on those bases. The main idea is that the \( 2n \) pairs needed to model the IAA membership...
function are compressed into the sum of $m$ weighted basis functions, where $m \ll 2n$.

We consider two different types of basis sets: linguistic basis sets (LBS) and general basis sets (GBS). The method for each type of basis is formulated to ensure that the resulting compressed membership function has membership degrees on $[0, 1]$, and that the approximation error is minimized. The LBS representation puts a constraint on the basis function and weights, which ensures that the compressed membership function is on $[0, 1]$ throughout the entire real-valued range of the survey. This comes at a cost of the design of the LBS; i.e., not all basis sets are LBSs. With a GBS, one can use any set of basis functions on the range of the survey, but the resulting membership function is not guaranteed to be on $[0, 1]$ for the entire survey range. It will, however, be on $[0, 1]$ at the locations modeled by IAA: the $2n$ interval end-points in the survey responses. So, the GBS is flexible in that it can use any basis set, but in practice may need a clipping function to ensure that the compressed membership function returns values in $[0, 1]$. We now describe the compression process for each of LBS and GBS.

A. Linguistic Basis Functions

Consider a set of linguistic basis functions $\Lambda(x) = \{\lambda_1(x), \ldots, \lambda_m(x)\}$ on the range of the survey, as defined in Definition 1. The properties of $\Lambda$ are such that at any value $x$ in the survey domain, the sum of the basis functions equals 1. Since we will be modeling $\mu_A(x)$ by the weighted sum of this basis set, this ensures we can achieve a membership at any value in $[0, 1]$ in the survey domain.

**Definition 1** (Linguistic Basis Set). $\Lambda(x) = \{\lambda_1(x), \ldots, \lambda_m(x)\}$ is a set of basis functions, $\lambda_i : x \rightarrow [0, 1]$, such that

$$\sum_{i=1}^{m} \lambda_i(x) = 1, \forall x \in \text{survey domain}.$$ 

We call $\Lambda(x)$ a Linguistic Basis Set (LBS) as the individual basis functions $\lambda_i(x)$ represent linguistic elements of the survey range: e.g., ‘very low,’ ‘low,’ ‘medium,’ ‘high,’ ‘very high,’ and also, the set of all linguistic elements can be combined to form the core of the survey. Figure 5 shows an example of an LBS with 5 basis functions on the survey range $[0, 10]$.

Let $\mu_A(x)$ be represented as the weighted sum of an LBS,

$$\hat{\mu}_A(x; w) = \sum_{i=1}^{m} w_i \lambda_i(x), \quad (8)$$

where $w = (w_1, \ldots, w_m)^T$ is the vector of weights on the respective basis functions, $\lambda_i(x)$. If $w_i \in [0, 1]$, then it is easy to see that $0 \leq \hat{\mu}_A(x; w) \leq 1, \forall w$ and $\forall x$ in the survey domain. Hence, to get the “best” $\hat{\mu}_A(x; w)$ representation of $\mu_A(x)$, we wish to solve the squared-error objective,

$$w^* = \arg \min_w \frac{1}{2} \sum_i ||\mu_A(x_i) - \hat{\mu}_A(x_i; w)||^2, \quad w \in [0, 1]^m. \quad (9)$$

The squared-error objective at (9) is a linearly constrained least-squares problem, which can be solved by using Matlab’s `lsqnonlin` routine. However, for generalization purposes (and for later work we will propose on this problem), we now write (9) in its matrix-vector form. Let $x = (x_1, \ldots, x_{2n})^T$ be a vector of the set of values at which $\mu_A(x)$ is modeled by IAA.\(^2\) Let $u_A = (\mu_A(x_1), \ldots, \mu_A(x_{2n}))^T$ be the vector of the membership values of $A$ at $x$, and $u_A(w) = (\hat{\mu}_A(x_1; w), \ldots, \hat{\mu}_A(x_{2n}; w))^T$ is similarly defined. Finally, let $\Lambda = [\lambda_j(x_i)]^{2n \times m}$ be the matrix where the $i$th row and $j$th column is $\lambda_j(x_i)$. It is easy to see that (8) is $u_A(w) = \Lambda w$. Then the squared-error term of (9) can be rewritten as

$$\begin{align*}
\frac{1}{2} \sum_i ||\mu_A(x_i) - \hat{\mu}_A(x_i; w)||^2 &= \frac{1}{2} u_A^T u_A + \frac{1}{2} u_A(w)^T u_A(w) - u_A^T u_A(w) \\
&= \frac{1}{2} u_A^T u_A + \frac{1}{2} w^T \Lambda^T \Lambda w - u_A^T \Lambda w.
\end{align*}$$

Since $u_A^T u_A$ is a constant term, (9) can be rewritten as

$$w^* = \arg \min_w \frac{1}{2} w^T \Lambda^T \Lambda w - u_A^T \Lambda w, \quad w \in [0, 1]^m. \quad (10)$$

Equation (10) is a standard linear-constraint quadratic programming (QP) problem,

$$\arg \min_w \frac{1}{2} w^T H w + f^T w, \quad C w \leq b, \quad (11)$$

which can be solved by nearly any available QP solver, such as `quadprog` in Matlab. To express (10) as (11), $H = \Lambda^T \Lambda$, $f = -\Lambda^T u_A$, $C = (1_m, -1_m)^T$, and $b = (1_m, 0_m)^T$.

\(^2\)Note that in practice if you start with $n$ intervals, there are (at most) $2n$ values of $\mu_A(x)$ that are modeled by IAA.
### B. General Basis Functions

Consider a set of general basis functions \( \Phi(x) = \{\phi_1(x), \ldots, \phi_m(x)\} \) on the survey domain, where \( \phi_i : \mathbb{R} \rightarrow \mathbb{R} \). Unlike an LBS described in Definition 1, this set of basis functions follows no specified definition other than mapping \( x \) to the real number line. Let \( \mu_A(x) \) be represented as the weighted sum of \( \Phi(x) \),

\[
\hat{\mu}_A(x; w) = \sum_{i=1}^{m} w_i \phi_i(x),
\]

where \( w = (w_1, \ldots, w_m)^T \) is the vector of weights on the respective basis functions, \( \phi_i(x) \). Rather than constraining \( w \) directly to ensure that \( \hat{\mu}_A(x; w) \) has membership degrees on \([0,1]\), we did with the LBS, the constraint for the GBS representation will be applied in the squared-error objective.

Again, consider the vector representations defined earlier, \( x, u_A, \) and \( \hat{u}_A \), and let \( \Phi = [\phi_j(x_i)]^{2n \times m} \) be the matrix where the \( i \)-th row and \( j \)-th column is \( \phi_j(x_i) \). Thus, (12) is \( \hat{u}_A = \Phi w \). Going through the same process as we did for the LBS, the squared-error objective can be written as

\[
w^* = \arg\min_w \frac{1}{2} w^T \Phi^T \Phi w - u_A^T \Phi w, \quad \Phi w \in [0,1]^{2n},
\]

where the constraint now is on \( \hat{u}_A = \Phi w \) to be on the interval \([0,1]\). The objective function at (13) can be expressed as either a linear-constraint least square problem or as the QP at (11) by \( H = \Phi^T \Phi, \quad f = -\Phi^T u_A, \quad b = (1^T_m, 0^T_m)^T, \) and \( C = [\Phi^T - \Phi^T]^T \).

### C. Discussion

The LBS weights, given that they are on the interval \([0,1]\), not only represent a FS, but also are themselves FSs that represent the membership in each of the linguistic basis functions. Hence, if the LBS can be described by linguistic terms, such as ‘big’ or ‘small,’ then the weights represent the contribution of that linguistic term in the FS representing the survey responses. Section V demonstrates this for a survey where respondents were asked to indicate the interval between 0 and 10 that represents the word ‘little.’

Another attractive aspect of the basis function representation—either LBS or GBS—is that FS operations can be directly applied to the basis functions. Consider two FSs, \( A \) and \( B \), represented on the basis shown in Fig. 6, \( A \rightarrow w_A = (0.1, 0.6, 0.4, 0.05, 0.01)^T \) and \( B \rightarrow w_B = (0.5, 0.2, 0.1, 0.01, 0.03)^T \); illustrated in Fig. 6(a). The intersection of these two sets is shown in view (b), which is calculated by

\[
(A \cap B)(z) = \min \left\{ \sum_{i=1}^{5} (w_A)_i \phi_i(z), \sum_{i=1}^{5} (w_B)_i \phi_i(z) \right\}.
\]

The computational complexity of computing \( A \ast B \) using the basis functions is \( O(m) \), where, to compute \( A \ast B \) using the original IAA representation is \( O(n_A + n_B) \), which is the cost of sorting \( z \) into the IAA set of ordinals \( X \) and \( Y \)—assuming that there are \( n_A \) intervals in \( A \) and \( n_B \) in \( B \). Since

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Fig. 6: Example of fuzzy intersection applied to FSs \( A \) and \( B \) using the LBS representation.

Fig. 7: (a) Survey responses for linguistic meaning of ‘little’ \([7]\). Each horizontal blue line indicates a survey response. (b) IAA and eIAA results for ‘little’ survey responses.

\( m \ll (n_A + n_B) \), the basis function representation is more efficient.

For IT2 FSs, which are the result of IAA with uncertain interval inputs, the IT2 FS is completely defined by its UMF and LMF; hence, eIAA can simply be used on each of these two defining sets. Thus, this makes eIAA also useful for the case of uncertain interval inputs.

### V. Demonstration

Figure 7(a) shows 174 survey responses where the respondents were asked to indicate the interval for ‘little’ on a scale of 0-10 (see [7] for a full description of these data). Clearly, most respondents indicated ‘little’ as being on the low end of the scale. The blue line in Fig. 7(b) shows the IAA model and the dotted lines indicate the eIAA models of the 174 interval responses. The eIAA membership functions were built with 5, 7, and 15 basis functions. The weighted basis functions are shown in Fig. 8. As expected, as the number of basis functions is increased, the eIAA approximation of the IAA membership function is improved. The basis functions in Fig. 8 also show that the linguistic content of the survey responses favors basis functions that peak around 1-2, which is intuitively pleasing.

Next, we demonstrate how the eIAA method can be used for FS operations on the FSs produced by IAA. Figure 9(a) shows the FS produced by IAA on survey responses for the word ‘large.’ As expected, the high membership area for the IAA ‘large’ FS is around 8-9 on the survey scale. View (b) of Fig. 9 shows the result of the union of ‘little’ and ‘large’
The IAA method provides aggregation and analysis for crisp and uncertain interval survey responses. A challenge with IAA is that the resulting FSs are defined by a large number of membership-domain pairs—equal to the number of input survey responses. This can not only result in storage issues for huge survey pools, but also present challenges when operating on the resulting FSs produced by IAA. The proposed eIAA algorithm can significantly reduce the necessary storage of the FSs and computational complexity of FS operations. Furthermore, eIAA provides a way to approximate the survey responses with linguistic prototypes, i.e., linguistic basis functions, giving meaning to the IAA and survey analysis results. We demonstrated this on a real survey where respondents were asked to quantify the interval that describes the word ‘little.’ As seen in the demonstration of using IAA and eIAA with the words survey data, there were many survey respondents that selected intervals for these words that would be outside the typical definition of the word, i.e., most reasonable respondents would not choose a value of 10 for the word, ‘little.’ Some respondents also chose very wide intervals, with a few respondents choosing the interval 0-10 for their response. The IA and EIA approaches attack this problem with a statistical preprocessing and aggregation scheme. In the future, for cases where outlier removal is appropriate, we will examine using \( \ell_1 \) regularization with (9) and (10), which will prefer sparse solutions for the basis function weights—essentially, choosing the linguistic bases that best fit the responses and disregarding all others. We will also investigate how regularization can automate the trade-off between the number of basis functions and the accuracy of eIAA.

Lastly, we will investigate efficient methods for handling inter-expert or inter-source uncertainty in surveys, which in [8] were shown to be best represented by zSlice GT2 FSs. This can be accomplished by using the eIAA idea on each slice of the zSlice GT2 FS.

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REFERENCES