

Generalization of the Fuzzy Integral for Discontinuous Interval- and Non-Convex Interval Fuzzy Set-Valued Inputs

Christian Wagner

Horizon Digital Economy Research Institute
University of Nottingham
School of Computer Science
Nottingham, UK
Email: christian.wagner@nottingham.ac.uk

Derek T. Anderson

Mississippi State University
Electrical and Computer Engineering
Mississippi State, MS, USA
Email: anderson@ece.msstate.edu

Timothy C. Havens

Michigan Technological University
Electrical and Computer Engineering
Computer Science
Houghton, MI, USA
Email: thavens@mtu.edu

Abstract—The *Fuzzy Integral* (FI) is a powerful approach for non-linear data aggregation. It has been used in many settings to combine evidence (typically objective) with the known “worth” (typically subjective) of each data source, where the latter is encoded in a *Fuzzy Measure* (FM). While initially developed for the case of numeric evidence (integrand) and numeric FM, Grabisch et al. extended the FI to the cases of continuous intervals and normal, convex fuzzy sets (i.e., fuzzy numbers). However, in many real-world applications, e.g., explosive hazard detection based on multi-sensor and/or multi-feature fusion, agreement based modeling of survey data, anthropology and forensic science, or computing with respect to linguistic descriptions of spatial relations from sensor data, discontinuous interval and/or non-convex fuzzy set data may arise. The problem is no theory and algorithm currently exists for calculating the FI for such a case. Herein, we propose an extension of the FI to discontinuous interval- and convex normal *Interval Fuzzy Set* (IFS)-valued integrands (with a numeric FM). Our approach arises naturally from analysis of the Extension Principle. Further, we provide a computationally efficient approach to computing the proposed extension based on the union of the FIs on the combinations of continuous sub-intervals and we demonstrate the approach using examples for both the *Choquet FI* (CFI) and *Sugeno FI* (SFI).

Index Terms—fuzzy integral; discontinuous; non-convex; interval valued integrand; interval fuzzy set-valued integrand; extension principle

I. INTRODUCTION

Fuzzy integrals (FIs) have been widely used as a non-linear data fusion operator which aggregates commonly “objective” evidence (e.g., sensor readings) with a commonly subjective “worth” of subsets of information sources (e.g., sensor confidence). The worth of the different subsets (i.e., combinations) of data sources in this context is modelled as a *fuzzy measure* (FM). In practical applications, FMs are generally either specified by experts (common when the number of information sources is relatively small, e.g., 3 to 5) or learnt (e.g., genetic algorithms [1], quadratic programming [2], or gradient descent-based [3]).

Traditionally, most applications have relied on both numeric integrand and numeric FM. However, a few works have appeared regarding the generalization of both the integrand

as well as the FM with respect to *intervals* and *Fuzzy Sets* (FSs) [2, 4–11] or more precisely, *Fuzzy Numbers* (FNs), i.e., convex and normal type-1 FSs. In [12] and [13, 14], Anderson et al. introduced indirect and direct extensions of the FI for the case of convex non-normal FS-valued integrands.

In this work, we propose an extension of the FI for the cases of *discontinuous interval* and *non-convex normal interval FS*-valued integrands (using a numeric FM). We discuss our motivation for the proposed extension and discuss application scenarios. Further, we provide the rationale for the proposed approach, together with the mathematical detail and algorithms to compute the extended FI. Finally, we demonstrate our methods using a series of examples and discuss the results.

The remainder of this article is organized as follows. First, we describe our motivation and the wider context for developing an extension to the discontinuous/non-convex cases of the FI in Section II, followed by a review of the Sugeno and Choquet number and interval/FN-based FIs in Section III. Next, we investigate a generalization of the interval and FN-based FIs to discontinuous intervals and non-convex normal interval FSs in Section IV. Section V provides numeric examples and a discussion and is followed by conclusions in Section VI.

II. MOTIVATION

Previous extensions of the FI for number-valued integrands to continuous interval or FN-valued integrands have broadened the applicability of FIs and FMs in real world applications. Recently, we extended the case of FN-valued integrands to convex, non-normal FS-valued integrands, enabling fusion in real world contexts such as forensic applications in anthropology and criminal justice where non-normal FS inputs are a reality [12–14]. Table II is an overview of key papers in the literature, a non-exhaustive list, that is focused on extending the FI and FM.

The extensions proposed here of the FI to discontinuous intervals and non-convex normal interval FS-valued inputs is motivated by a similar need in real world applications. In part, this work is driven by a need in forensic science

TABLE I
ACRONYMS AND NOTATION (IN ALPHABETIC ORDER)

CFI	Choquet fuzzy integral
SFI	Sugeo fuzzy integral
EP	Extension Principle
FI	Fuzzy integral
FM	Fuzzy measure
FN	Fuzzy number, i.e., a convex, normal fuzzy set
FS	Fuzzy set
IFS	Interval fuzzy set, e.g., A , where $\mu_A \in \{0, 1\}$
$FN([0,1])$	the set of FNs on $[0,1]$
g	real-valued FM, $g : 2^X \rightarrow [0, 1]$
h	real-valued evidence from X , $h : X \rightarrow \mathfrak{R}$
h_i	real-valued evidence from x_i
\bar{h}_i	continuous I -valued evidence from x_i
$\bar{\bar{h}}_i$	discontinuous I -valued evidence from x_i
H_i	FN-valued evidence from x_i
\bar{H}_i	convex, normal interval-FS (i.e., interval FN) valued evidence from x_i
$\bar{\bar{H}}_i$	non-convex, normal interval-FS valued evidence from x_i
I	set of intervals, $\{\bar{u} \subseteq \mathfrak{R} : \bar{u} = [u^l, u^r], u^l \leq u^r\}$
X	set of sources (sensors, experts, etc.)
x_i	i th source ($x_i \in X$)

[13, 14] to fuse complicated non-convex FS-valued data. Additional examples of applications where this applies range from agreement-based modeling of survey data [15] to explosive hazard detection (e.g., [6]), computing with respect to linguistic descriptions of spatial relations of sensor data generated from the histograms-of-forces (e.g., [16–18]), and even sensor measurement-based histograms [19]. Furthermore, we envisage three future applications of the proposed extension:

- 1) Further generalization to non-convex, normal FS-valued (i.e. not interval FS valued) integrands.
- 2) Full generalization to the case of non-convex, non-normal FS-valued and eventually, type-2 FS-valued integrands.
- 3) Aggregation of output FSs from a series of type-1 fuzzy logic systems;

While the traditional approach in applications where non-convex inputs are present has been to force convexity [6, 19], a direct approach promises not only greater accuracy but also mitigates the need for additional post-processing of the raw data. We also consider the non-convexity property of the inputs (and, most likely, the outputs) to have possible importance in certain applications, e.g., where a bi-modal distribution of the FS is informative. As alluded to above, it is clear that, eventually, a full generalization to non-convex, non-normal FS inputs is desirable. We are planning to address this as a future publication through a combination of our work in this paper and our extensions to non-normal FS-valued inputs in [12–14].

III. THE SUGENO AND CHOQUET FUZZY INTEGRALS

The fusion of information using the *classical* (i.e., real-valued integrand and FM) *Sugeno FI* (SFI) and *Choquet FI* (CFI) has a rich history. Much of the theory and several applications can be found in [4, 20]. With respect to this problem, we consider a finite set of sources of information $X = \{x_1, \dots, x_N\}$ and a function that maps X into some

domain, commonly $[0, 1]$, that represents the partial support of a hypothesis from the standpoint of each source of information. Depending on the problem domain, X can be a set of experts, sensors, features, pattern recognition algorithms, etc. For example, in our prior work, X was a set of methods that help determine the age-at-death of a person from their skeletal remains [13, 14], or input from a series of crowd-sourced ratings about restaurant quality [19]. Both SFI and CFI take (typically objective) partial support for the hypothesis from the standpoint of each source of information and fuse it with the (perhaps subjective) worth (or reliability) of each subset of X in a non-linear fashion. This worth is encoded in a FM [21]. We now review different extensions of the FI, specifically in terms of the nature of the integrand values employed (while maintaining a number-valued FM).

A. The Real-Valued FI

Initial definitions focused on $h : X \rightarrow [0, 1]$ and FM $g : 2^X \rightarrow [0, 1]$. These can be defined more generally, and have been. For example, it is convenient to think of the function h and output of FM g on the unit interval for scenarios such as confidence fusion (we will stick to this convention for the FM g). However, we define the function h more generally as $h : X \rightarrow \mathfrak{R}$, where h can now be thought of directly as inputs such as sensor readings.¹

More formally, for a finite set X , a FM is a function $g : 2^X \rightarrow [0, 1]$, such that

1. $g(\emptyset) = 0$ and $g(X) = 1$;
2. If $A, B \subseteq X$ with $A \subseteq B$, then $g(A) \leq g(B)$.

Note, if X is an infinite set, a third condition guaranteeing continuity is required, but this is a moot point for finite X as considered in this paper and most practical applications. Given a finite set X , a FM g and a function $h : X \rightarrow \mathfrak{R}$, the (real-valued) SFI of h with respect to g is

$$\int_S h \circ g = \bigvee_{i=1}^N h_{\pi(i)} \wedge g(A_i), \quad (1)$$

while the CFI of h with respect to g is

$$\int_C h \circ g = \sum_{i=1}^N h_{\pi(i)} (g(A_{(i)}) - g(A_{(i-1)})), \quad (2)$$

where π is a permutation on X , such that $h_{\pi(1)} \geq h_{\pi(2)} \geq \dots \geq h_{\pi(N)}$, $A_{(i)} = \{x_{\pi(1)}, \dots, x_{\pi(i)}\}$ and $g(A_{(0)}) = 0$ [2, 21].

B. The Interval- and Fuzzy Number-Valued FI

Let $\bar{h}(x_i) \subseteq I$ be the continuous interval-valued evidence from source x_i where $I = \{\bar{u} \subseteq \mathfrak{R} : \bar{u} = [u^l, u^r], u^l \leq u^r\}$ is the set of all real-valued continuous intervals.²

¹We do note that this would likely effect the utilization of the SFI, as the measure and integrand would likely reside at different scales, perhaps negatively affecting the results of the max and min operators. However, this does not impact the CFI in the same way, that is, mathematically.

²Again, in most practical circumstances, e.g., confidence level fusion, \bar{h} is constrained to be continuous interval subsets of the unit-interval. However, without loss of generality, we consider the more general case of \mathfrak{R} .

TABLE II
SELECTED WORKS ON THE EXTENSION OF THE FI AND FM FOR DIFFERENT TYPES OF INPUTS

		Integrand			
		Numeric	Interval		Fuzzy Set
			Continuous	Discontinuous	
FM	Numeric	[2, 4, 20, 21]	[2]	<i>this paper</i>	Convex, normal: [2, 6, 22] Convex, non-normal: [12, 13] Non-convex, normal: <i>this paper (for IFS)</i> Type-II FNs: [23–25]
	Interval				
	Continuous	[7]	[7]		[7]
	Discontinuous				
	Fuzzy Set	Convex, normal: [7]	Convex, normal: [7]		Convex, normal: [7, 26]

In [2], Grabisch et al. put forth continuous interval- and FN-valued generalizations of the FI based in part on the interval and fuzzy arithmetic work of Dubois and Prade [5]. Dubois and Prade [5] showed the following.

Theorem 1. If a function φ is continuous and non-decreasing, then, when defined on continuous intervals, it produces a continuous interval $\varphi(\bar{u}) = [\varphi(u^l), \varphi(u^r)]$.

They extended their interval proofs and formed an α -cut based definition for normal convex FSs, i.e., FNs (adopting a decomposition theorem approach that is a direct result of the EP). The interval approach is of particular benefit as it provides a computationally efficient algorithmic basis for performing fuzzy arithmetic and the FI.

The CFI and SFI are idempotent, continuous and monotonically non-decreasing functions [4]. Grabisch et al. leveraged these properties and Dubois's and Prade's work in order to extend the CFI and SFI to continuous interval-valued integrands.

Definition 1. (FI with interval-valued integrand) Let $\bar{h} : X \rightarrow I$ denote the interval-valued partial support function and let $\bar{h}_i = [\bar{h}_i^l, \bar{h}_i^r]$ denote the i^{th} interval (where \bar{h}_i^l and \bar{h}_i^r are the left and right interval endpoints respectively). The interval-valued FI is [4]

$$\int \bar{h} \circ g = \left[\int \bar{h}^l \circ g, \int \bar{h}^r \circ g \right]. \quad (3)$$

Definition 2. (FI with FN-valued integrands) Let $H : X \rightarrow FN(\mathfrak{R})$ denote the FN-valued partial support function and H_i the i^{th} FN. Additionally, let $[H_i]_\alpha = [(h_i)_\alpha^l, (h_i)_\alpha^r]$ for $0 \leq \alpha \leq 1$. The FI for FN-valued integrands is [4]

$$\int H \circ g = \bigcup_{\alpha \in [0,1]} \alpha \left[\int H \circ g \right]_\alpha \quad (4)$$

$$= \bigcup_{\alpha \in [0,1]} \alpha \left[\int [H]_\alpha \circ g \right], \quad (5)$$

These formulations of the FI follow directly from Sugeno and Dubois and Prade's works [5, 21]. In addition, Sugeno showed the following properties for the FI with real-valued integrands (h) and numeric FMs (g).

Property 1. (Continuity of $\int h \circ g$) $\int h \circ g$ is continuous.

Property 2. (Boundedness of $\int h \circ g$) $\int h \circ g$ is bounded between $h_1 \wedge \dots \wedge h_N$ and $h_1 \vee \dots \vee h_N$.

Further, Dubois and Prade [5] showed the following properties for the FI with interval-valued integrands (\bar{h}) and numeric FMs (g).

Property 3. (Continuity of $\int \bar{h} \circ g$) $\int \bar{h} \circ g$ produces a continuous interval.

Property 4. (Boundedness of $\int \bar{h} \circ g$) $\int \bar{h} \circ g$ produces an interval $[a, b]$, $a, b \in [0, 1]$, where a is bounded between $h_1^l \wedge \dots \wedge h_N^l$ and $h_1^l \vee \dots \vee h_N^l$ and b is bounded between $h_1^r \wedge \dots \wedge h_N^r$ and $h_1^r \vee \dots \vee h_N^r$.

The FIs defined at Eq. (3-5) are appropriate for continuous intervals and FNs, but are not for discontinuous intervals or non-convex interval FSs, which is where we turn to next.

IV. BEYOND CONTINUOUS INTERVALS

In this section, we propose an extension of the FI to discontinuous intervals and from this, to normal, non-convex *interval FSs* (IFS). Intuitively, an extension to discontinuous intervals should arise directly from the EP. Let $\mathbf{z} = (z_1, \dots, z_N)$ be a vector of numbers which produce the set

$$S_y = \left\{ \mathbf{z} : \mathbf{z} \in \mathfrak{R}^N, \int \mathbf{z} \circ g = y \right\}. \quad (6)$$

The set S_y is all admissible N -tuples of values such that the real-valued FI of \mathbf{z} , given FM g , maps into a given value y . Then, for FS-valued integrands H we have

$$\left(\int H \circ g \right) (y) = \bigvee_{\mathbf{z} \in S_y} \{H_1(z_1) \wedge \dots \wedge H_N(z_N)\}. \quad (7)$$

From Eq. (7), it is clear that according to the EP, the FI is the maximum of the minimums of the corresponding membership degrees over all admissible solutions at a given y .

Since the focus of this article is the extension of the FI to the case of discontinuous intervals, we review the concept of the IFS. The extension to the case of discontinuous intervals is simplified if one represents \bar{h} as a convex, normal IFS, which is, more formally, defined below.

Definition 3. (Mapping of \bar{h} to IFS \bar{H}) An IFS \bar{H}_i is a convex, normal (i.e., height = 1) FS $\{\bar{H}_i(z) \in \{0, 1\} : z \in \mathfrak{R}\}$

such that

$$\bar{H}_i(z) = \begin{cases} 1, & z \in \bar{h}_i, \\ 0, & \text{else.} \end{cases}$$

Remark 1. For all intents and purposes, \bar{H}_i and \bar{h}_i are equivalent representations of the interval \bar{h}_i . \bar{H}_i is simply a FS version of \bar{h}_i that we will use to apply the EP.

We can now apply the EP to express the FI with an \bar{H} -valued integrand, i.e. $\int \bar{H} \circ g$. Because $\bar{H}_i(z) \in \{0, 1\}$, the expression at Eq. (7) reduces to

$$\left(\int \bar{H} \circ g \right) (y) = \begin{cases} 1, & \exists \mathbf{z} \in S_y : \bigwedge_{i=1}^N \bar{H}_i(z_i) = 1, \\ 0, & \text{else.} \end{cases} \quad (8)$$

This equation can be further reduced by using the interval notation $\bar{h}_i = [\bar{h}_i^l, \bar{h}_i^r]$,

$$\left(\int \bar{H} \circ g \right) (y) = \begin{cases} 1, & \exists \mathbf{z} \in S_y : \bar{h}_i^l \leq z_i \leq \bar{h}_i^r, \forall i, \\ 0, & \text{else.} \end{cases} \quad (9)$$

Remark 2. Equation (9) is a direct result of the EP and the mapping of \bar{h} to \bar{H} . This equation shows that the FI with an \bar{H} -valued integrand is nothing more than an ‘‘inclusion check’’ at each y for the existence of any admissible \mathbf{z} that satisfies $\int \mathbf{z} \circ g = y$, such that $\bar{h}_i^l \leq z_i \leq \bar{h}_i^r, i = 1, \dots, N$.

Proposition 2. The FI at Eq. (9) leads directly to the interval FI proposed by Grabisch et al. at Eq. (3).

Proof: The function $\int \mathbf{z} \circ g$ is monotonic and non-decreasing. Hence, because $(\int \bar{H} \circ g)(y) = 1$ iff there exists a $\mathbf{z} \in S_y$ such that $\bar{h}_i^l \leq z_i \leq \bar{h}_i^r, \forall i$, then for any admissible \mathbf{z} ,

$$y^l = \int \bar{h}^l \circ g \leq \int \mathbf{z} \circ g \leq \int \bar{h}^r \circ g = y^r.$$

Hence, $(\int \bar{H} \circ g)(y) = 1$ *only* on the interval $[y^l, y^r]$, which implies Eq. (3). ■

A. FI for discontinuous intervals

We now move on to proposing a FI for discontinuous intervals. First, let

$$\bar{h}_i = \bigcup_{j=1}^{M_i} [\bar{h}_i]_j, \quad (10)$$

be the discontinuous interval-valued evidence from source x_i where M_i is the number of continuous (sub-)intervals $[\bar{h}_i]_j$ which make up the overall discontinuous interval \bar{h}_i . This formulation of a discontinuous interval will be important to our definition of the FI for discontinuous interval integrands. To extend \bar{h} to $\bar{\bar{h}}$ and \bar{H} to $\bar{\bar{H}}$, we first use the following lemma.

Lemma 3. (Extension of \bar{h} to $\bar{\bar{h}}$ and \bar{H} to $\bar{\bar{H}}$) For each $[\bar{h}_i]_j$, let

$$Z_{ji} = \{z \in \mathfrak{R} : z \in [\bar{h}_i]_j\}.$$

Furthermore, let

$$Z_i = \bigcup_{j=1}^{M_i} Z_{ji},$$

that is, all $z \in \mathfrak{R}$ that are in $\bar{\bar{h}}_i$, where $\bar{\bar{h}}_i$ is computed at Eq. (10). Therefore, Z_i makes up the support of $\bar{\bar{H}}_i$. That is,

$$\bar{\bar{H}}_i(z) = \begin{cases} 1, & z \in Z_i, \\ 0, & \text{else.} \end{cases} \quad (11)$$

Because we have written the discontinuous intervals $\bar{\bar{h}}$ as FSs $\bar{\bar{H}}$, it is easy to now apply the EP to define a FI for discontinuous intervals.

Definition 4. (FI for $\bar{\bar{h}}$) The FI for $\bar{\bar{h}}$ is

$$\left(\int \bar{\bar{H}} \circ g \right) (y) = \begin{cases} 1, & \exists \mathbf{z} \in S_y : z_i \in Z_i, \forall i, \\ 0, & \text{else.} \end{cases} \quad (12)$$

Thus,

$$\int \bar{\bar{h}} \circ g = \left\{ y \in \mathfrak{R} : \left(\int \bar{\bar{H}} \circ g \right) (y) = 1 \right\}. \quad (13)$$

Remark 3. Our definition of the FI for discontinuous intervals at Eq. (13) is derived directly from the EP; hence, it is theoretically valid. However, Eq. (13) does not provide a computationally attractive solution as do the FIs at Eq. (3) and Eq. (5), and at our EP-derived interval FI at Eq. (9). But, as will be shown below, we can express Eq. (13) as the union of the FIs of numbers, much in the way that Eq. (3) and Eq. (5) also do.

We now propose a computationally-tractable solution to the FI for discontinuous intervals.

Theorem 4. The FI of discontinuous interval-valued integrand $\bar{\bar{h}}$ with respect to the numeric FM g can be computed as

$$\begin{aligned} \int \bar{\bar{h}} \circ g &= \bigcup_{k=1}^{\mathcal{M}} \int [\bar{h}]_k \circ g, \\ &= \bigcup_{k=1}^{\mathcal{M}} \left[\int [\bar{h}]_k^l \circ g, \int [\bar{h}]_k^r \circ g \right], \end{aligned} \quad (14)$$

where $[\bar{h}]_k$ is the k th N -tuple of the power set of all sub-intervals in \bar{h} and $\mathcal{M} = \prod_{i=1}^N M_i$; e.g., $[\bar{h}]_1 = \{[\bar{h}_1]_1, \dots, [\bar{h}_N]_1\}$ and $[\bar{h}]_{\mathcal{M}} = \{[\bar{h}_1]_{M_1}, \dots, [\bar{h}_N]_{M_N}\}$.

Proof: Let Z_i be the sets defined in Lemma 3. The set of possible $z \in S_y : z_i \in Z_i, \forall i$, in Eq. (12) can be expressed as

$$\begin{aligned} \mathbf{Z}_y &= \left\{ \mathbf{z} \in \mathfrak{R}^N : z_i \in Z_i, \int \mathbf{z} \circ g = y \right\} \\ &= \left\{ \mathbf{z} \in \mathfrak{R}^N : z_i \in \bigcup_{j=1}^{M_i} Z_{ji}, \int \mathbf{z} \circ g = y \right\}. \end{aligned} \quad (16)$$

By distributing the union, we can reformulate \mathbf{Z}_y as

$$\mathbf{Z}_y = \bigcup_{k=1}^{\mathcal{M}} \left\{ \mathbf{z} \in \mathfrak{R}^N : z_i \in Z_{ki}, \int \mathbf{z} \circ g = y \right\}. \quad (17)$$

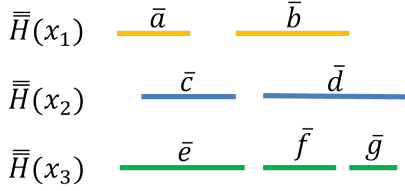


Fig. 1. Example for discontinuous $I([0, 1])$ -valued partial support function.

where \mathcal{Z}_k is the k th tuple of the power set of Z_i s, viz., $\mathcal{Z}_1 = \{Z_{11}, \dots, Z_{1N}\}$ and $\mathcal{Z}_{\mathcal{M}} = \{Z_{M_1 1}, \dots, Z_{M_N N}\}$. Let \mathbf{Z}_{y_k} be the k th term in the union in Eq. (17), where $\mathbf{Z}_y = \bigcup_{k=1}^{\mathcal{M}} \mathbf{Z}_{y_k}$, then Eq. (12) can be written as

$$\left(\int \bar{H} \circ g \right) (y) = \begin{cases} 1, & \exists \mathbf{z} \in \bigcup_{k=1}^{\mathcal{M}} \mathbf{Z}_{y_k}, \\ 0, & \text{else.} \end{cases} \quad (18)$$

Let

$$\left(\int (\bar{H})_k \circ g \right) (y) = \begin{cases} 1, & \exists \mathbf{z} \in \mathbf{Z}_{y_k}, \\ 0, & \text{else,} \end{cases} \quad (19)$$

where this is the standard FS-integrand FI of the k th tuple of subset components of \bar{H} , viz., $\bar{H} = \bigcup_{k=1}^{\mathcal{M}} (\bar{H})_k$, where the support of $(\bar{H})_k$ is \mathcal{Z}_k . Combining Eq. (18) and Eq. (19) gives

$$\left(\int \bar{H} \circ g \right) (y) = \bigcup_{k=1}^{\mathcal{M}} \left(\int (\bar{H})_k \circ g \right) (y),$$

which combined with the result of Proposition 2 proves the theorem. \blacksquare

Remark 4. The advantage of our formulation of the FI for discontinuous intervals at Eq. (15) is that it is simply the union of all the combinations of continuous-valued results. Moreover, since $(\int [\bar{h}]_k \circ g) = [\int [\bar{h}]_k^l \circ g, \int [\bar{h}]_k^r \circ g]$, each continuous-interval FI is (a) characterized by the FI on the interval endpoints and (b) continuous on the interval (albeit, on the continuous-interval sub-parts of \bar{h}). This allows for the efficient calculation of $\bar{h} \circ g$ in terms of just the union of the resulting continuous closed intervals, which only require the real-valued FI to be calculated on the interval endpoints. Obviously, this concept holds for the corresponding case of $\int \bar{H} \circ g$.

Remark 5. Our definition of the discontinuous-interval FI reduces to the existing form of the FI for continuous intervals and FNs as the union-based decomposition results in a set \mathcal{Z} of size one, viz. $\mathcal{M} = 1$.

Example 1. Consider the evidence from three interval-fuzzy-set-valued information sources as shown in Fig. 1. By Theorem 4, we know that in order to compute the FI over all three information sources, we need to consider all possible combinations (z_1, z_2, z_3) , where $z_1 \in \{\bar{a}, \bar{b}\}$, $z_2 \in \{\bar{c}, \bar{d}\}$ and $z_3 \in \{\bar{e}, \bar{f}, \bar{g}\}$. Further, we know that for each of these \mathcal{M} combinations, $\{\{\bar{a}, \bar{c}, \bar{e}\}_1, \dots, \{\bar{b}, \bar{d}, \bar{g}\}_\mathcal{M}\}$, $\int (\bar{H})_{k \in \{1, \dots, \mathcal{M}\}} \circ g$ is a closed continuous interval.

TABLE III
INTERVAL-VALUED INPUTS FOR NUMERIC EXAMPLES

Evidence #	a_1^l	a_1^r	a_2^l	a_2^r	b_1^l	b_1^r	b_2^l	b_2^r
Continuous Intervals (Figs. 2, 3)	0.1	0.7			0.2	0.8		
Discontinuous Intervals (Figs. 4, 5, 6, 7)	0.1	0.5	0.6	0.7	0.2	0.4	0.65	0.8

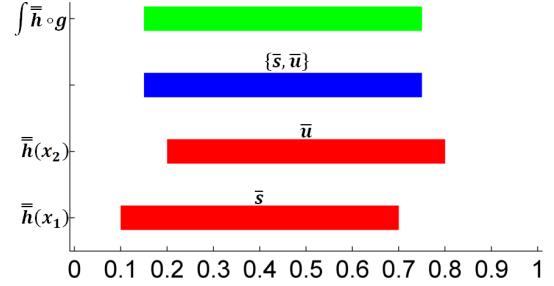


Fig. 2. Example of the CFI for two sources of evidence, x_1 and x_2 , where $g_{x_1} = g_{x_2} = 0.5$. As both sources provide continuous intervals, only one combination of “sub-components” is computed for the FI (i.e., $\{\bar{s}, \bar{u}\}$ in blue). The result of the FI is shown in green.

V. NUMERIC EXAMPLES

In this section, we provide illustrative examples of applying the FI at Eq. (15) to a variety of inputs. Namely, we show examples for continuous interval-valued and convex, normal IFS-valued integrands (mainly to show the recovery of the results from the original extension [2]). We then proceed to examples demonstrating the FI for discontinuous interval and non-convex, normal IFS-valued integrands. For all examples, both outputs generated using the CFI and SFI are shown and numeric FMs are used throughout. All examples (for both continuous and discontinuous intervals) are based on the interval-valued evidence included in Table III.

A. FI for interval-valued integrands

1) *Continuous intervals:* For the case of continuous intervals, Eq. (15) reverts to the FI introduced by Grabisch et al. [2] (see Remark 5). In the examples reported in Figs. 2 and 3 (CFI and SFI respectively), two sources of evidence provide continuous intervals, \bar{s} and \bar{u} , as evidence. These sources are aggregated using a FM of equal “worth.” As both sources provide continuous intervals as evidence, the FI reduces to the direct combination of both intervals, i.e., $\{\bar{s}, \bar{u}\}$, using Eq. (3).

2) *Discontinuous intervals:* In Figs. 4 and 5, two sources of evidence, x_1 and x_2 , which each provide discontinuous interval-valued evidence, are aggregated. The first case is for a FM with equal worth, i.e., $g_{x_1} = g_{x_2} = 0.5$, and the second case is for the FM $g_{x_1} = 0.2$ and $g_{x_2} = 0.8$. These examples show how the CFI on discontinuous intervals may lead to both continuous and discontinuous outputs depending on both the actual evidence and the FMs employed. Note how in both examples the FI on discontinuous intervals is broken up into four FIs on continuous subintervals (in blue) which

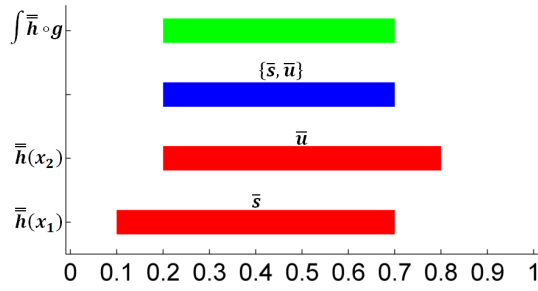


Fig. 3. Example of the SFI for two sources of evidence, x_1 and x_2 , where $g_{x_1} = 0.5$ and $g_{x_2} = 0.5$. As both sources provide continuous intervals, only one combination of “sub-components” is computed for the FI (i.e., $\{\bar{s}, \bar{u}\}$ in blue). The result of the FI is shown in green.

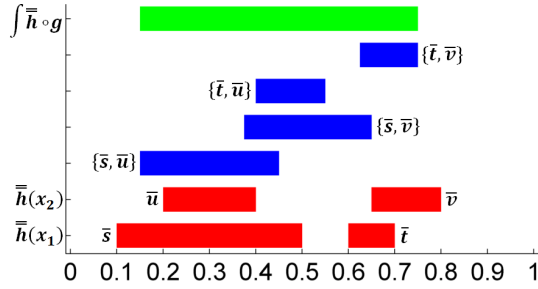


Fig. 4. Example of the CFI for two sources of evidence, x_1 and x_2 , where $g_{x_1} = g_{x_2} = 0.5$. As both sources provide discontinuous intervals, four combinations of sub-components are computed for the FI (i.e., $\{\bar{s}, \bar{u}\}$, $\{\bar{s}, \bar{v}\}$, $\{\bar{t}, \bar{u}\}$ and $\{\bar{t}, \bar{v}\}$ in blue). The result of the FI (based on the union of FIs of the combinations) is shown in green.

are subsequently combined into an overall output (in green) by using the union of the resulting continuous intervals. Figures 6 and 7 show the same example for the SFI. What is apparent is the difference between the SFI and the CFI. Namely, the CFI can produce any number between the *min* and *max* whereas the SFI is restricted to one of the input or lattice values. Therefore, one might assume that the SFI is more likely to produce a greater number of discontinuous intervals.

B. FI for normal IFS-valued integrands

Leading on from the interval-valued cases considered in the previous subsection, we now consider the cases of {convex,

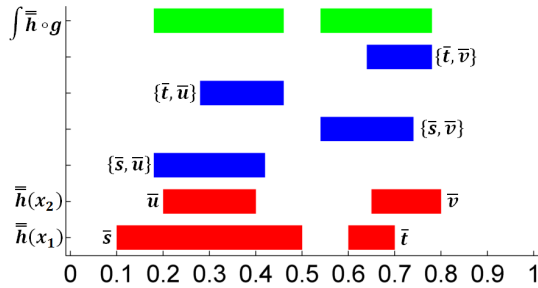


Fig. 5. Example of the CFI for two sources of evidence, x_1 and x_2 , where $g_{x_1} = 0.2$ and $g_{x_2} = 0.8$. As both sources provide discontinuous intervals, four combinations of sub-components are computed for the FI (i.e., $\{\bar{s}, \bar{u}\}$, $\{\bar{s}, \bar{v}\}$, $\{\bar{t}, \bar{u}\}$ and $\{\bar{t}, \bar{v}\}$ in blue). The result of the FI (based on the union of FIs of the combinations) is shown in green.

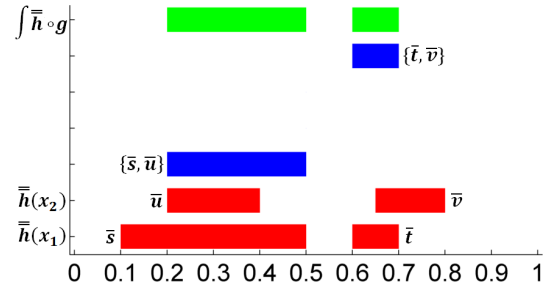


Fig. 6. Example of the SFI for two sources of evidence, x_1 and x_2 , where $g_{x_1} = g_{x_2} = 0.5$. As both sources provide discontinuous intervals, four combinations of sub-components are computed for the FI (i.e., $\{\bar{s}, \bar{u}\}$, $\{\bar{s}, \bar{v}\}$, $\{\bar{t}, \bar{u}\}$ and $\{\bar{t}, \bar{v}\}$ in blue). The result of the FI (based on the union of FIs of the combinations) is shown in green. Note, the FI on the combinations $\{\bar{s}, \bar{v}\}$ and $\{\bar{t}, \bar{u}\}$ both result in the singleton 0.5 which is why they are not visible on the figure.

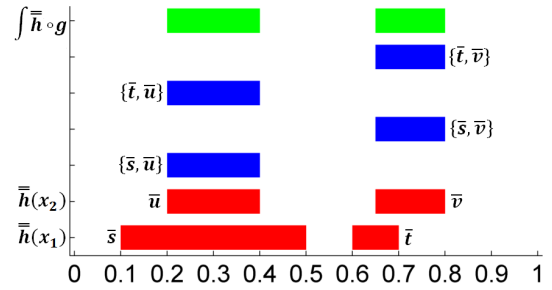


Fig. 7. Example of the SFI for two sources of evidence, x_1 and x_2 , where $g_{x_1} = 0.2$ and $g_{x_2} = 0.8$. As both sources provide discontinuous intervals, four combinations of sub-components are computed for the FI (i.e., $\{\bar{s}, \bar{u}\}$, $\{\bar{s}, \bar{v}\}$, $\{\bar{t}, \bar{u}\}$ and $\{\bar{t}, \bar{v}\}$ in blue). The result of the FI (based on the union of FIs of the combinations) is shown in green.

normal} and {non-convex, normal} IFS-valued integrands. The examples are directly based on the interval-valued FI examples in Section V-A.

1) *Convex, normal IFSs*: As discussed, for convex, normal IFS-valued integrands, Eq. (15) reverts to the FI for FN-valued integrands as introduced by Grabisch et al. [2] and shown in Eq. (5). Figures 8 and 9 (CFI and SFI, respectively) illustrate an example for two convex IFS-valued sources, x_1 and x_2 , where $g_{x_1} = g_{x_2} = 0.5$ (i.e., equal worth).

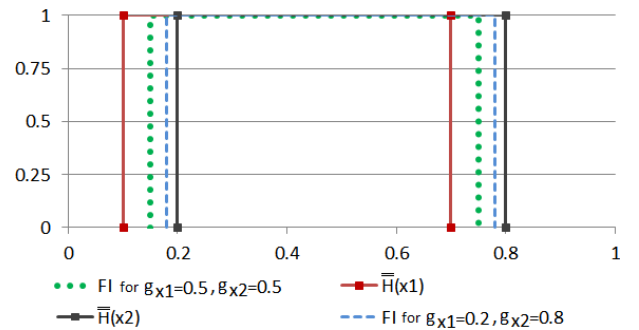


Fig. 8. Example of the CFI for two convex, normal IFS-valued sources of evidence, x_1 and x_2 , where $g_{x_1} = g_{x_2} = 0.5$.

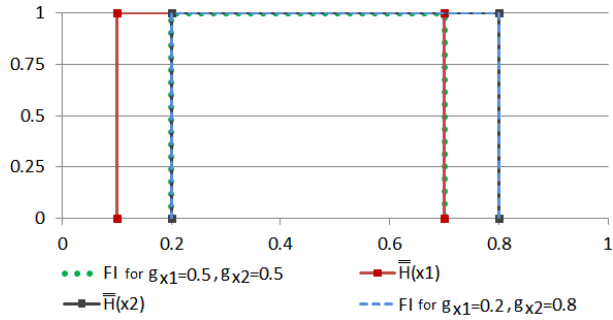


Fig. 9. Example of the SFI for two convex, normal IFS-valued sources of evidence, x_1 and x_2 , where $g_{x_1} = 0.5$ and $g_{x_2} = 0.5$.

2) *Non-convex, normal IFSs*: Figure 10 shows an example of the CFI for non-convex IFS-valued inputs, x_1 and x_2 , for two different FMs ($g_{x_1} = g_{x_2} = 0.5$ and also $g_{x_1} = 0.2$, $g_{x_2} = 0.8$, respectively). Figure 11 shows the same example for the case of the SFI.

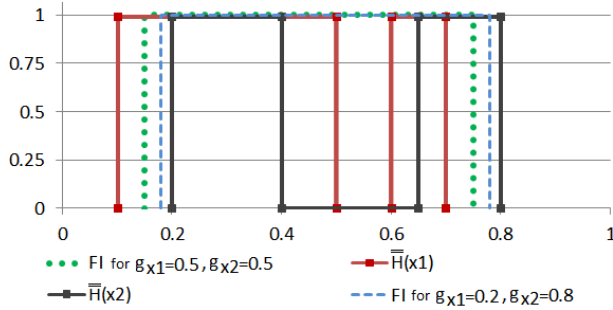


Fig. 10. Example of the CFI for two non-convex, normal IFS sources of evidence x_1 and x_2 for two different FMs.

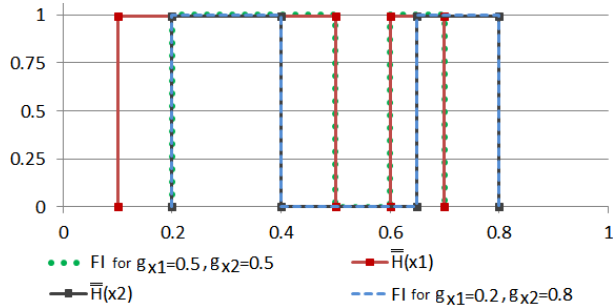


Fig. 11. Example of the SFI for two non-convex, normal IFS sources of evidence x_1 and x_2 for two different FMs.

VI. CONCLUSION

In this paper, we introduced an extension of the FI from continuous interval- and FN-valued integrands (and a numeric FM) to discontinuous interval- and convex normal IFS-valued integrands. We showed how the proposed extension arises naturally from the EP when one considers IFSs and that the resulting extension applies to both discontinuous intervals and

convex-normal IFSs. Having formally established the extension of the FI, we have provided a computationally efficient approach based on the union of the FIs of all combinations of continuous sub-intervals (which each can be computed based on the interval endpoints as shown by Grabisch et al. [2]).

We also demonstrated numeric examples for both the CFI and SFI for the cases of interval- and convex normal IFS-valued integrands and we demonstrated how the result of the FI can itself be continuous (convex) or discontinuous (non-convex) based on both the actual value of the integrands as well as the FM.

In future work, we will leverage the theorems in this article together with our previous work on non-normal integrand-valued FIs [12] to develop a powerful general case extension of the FI for any type of FS (convex/non-convex and normal/sub-normal) as well as type-2 FSs (and all set properties therein).

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