Fuzzy Set Theory in Computer Vision: **Example 10**

Derek T. Anderson and James M. Keller

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Shapley

The Shapley values of μ are

$$\Phi_\mu(i) = \sum_{K \subseteq X \backslash \{i\}} \zeta_{X,1}(K)\left(\mu(K \cup \{i\}) - \mu(K)\right), \tag{1a}$$

$$\zeta_{X,1}(K) = \frac{(|X| - |K| - 1)!|K|!}{|X|!}, \tag{1b}$$

where $K \subseteq X \backslash \{i\}$ denotes all proper subsets from X that do not include source i. The Shapley value of μ is the vector $\Phi_\mu = (\Phi_\mu(1), ..., \Phi_\mu(N))^t$ and $\sum_{i=1}^{N} \Phi_\mu(i) = 1$. The Shapley index can be interpreted as the average amount of *contribution* of source i across all coalitions. It makes its decision based on the weighted sum of (positive-valued) numeric differences between consecutive steps (layers) in the capacity.

Interaction index

The interaction index (Murofushi and Soneda) between i and j is

$$\mathbf{I}_{\mu}(i,j) = \sum_{K \subseteq X \setminus \{i,j\}} \zeta_{X,2}(K)(\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)), \quad i = 1, ..., N, \tag{2a}$$

$$\zeta_{X,2}(K) = \frac{(|X| - |K| - 2)!|K|!}{(|X| - 1)!}, \tag{2b}$$

where $\mathbf{I}_{\mu}(i,j) \in [-1,1], \forall i,j \in \{1,2,...,N\}$. A value of 1 (respectively, -1) represents the maximum complementarity (respective redundancy) between i and j.

Interaction index: generalization

The interaction index for any coalition $A \subseteq X$ is

$$\mathbf{I}_\mu(A) = \sum_{K \subseteq X \setminus A} \zeta_{X,3}(K, A) \sum_{C \subseteq A} (-1)^{|A \setminus C|} \mu(C \cup K),$$

$$i = 1, ..., N, \tag{3a}$$

$$\zeta_{X,3}(K, A) = \frac{(|X| - |K| - |A|)!|K|!}{(|X| - |A| + 1)!}. \tag{3b}$$

This equation is a generalization of both the Shapley index and Murofushi and Soneda's interaction index as $\Phi_\mu(i)$ corresponds with $\mathbf{I}_\mu(\{i\})$ and $\mathbf{I}_\mu(i,j)$ with $\mathbf{I}_\mu(\{i,j\})$.